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Abraham Wald's Work on Aircraft Survivability

MARC MANGEL and FRANCISCO J. SAMANIEGO*

While he was a member of the Statistical Research Group (SRG), Abraham Wald worked on the problem of estimating the vulnerability of aircraft, using data obtained from survivors. This work was published as a series of SRG memoranda and was used in World War II and in the wars in Korea and Vietnam. The memoranda were recently reissued by the Center for Naval Analyses. This article is a condensation and exposition of Wald's work, in which his ideas and methods are described. In the final section, his main results are reexamined in the light of classical statistical theory and more recent work.

KEY WORDS: Survivability; Missing data; Approximate methods; Maximum likelihood.

1. INTRODUCTION

December 7, 1981, was the 40th anniversary of the attack on Pearl Harbor, the subsequent entry of the United States into World War II, and also the birth of the Statistical Research Group (SRG) and the Antisubmarine Warfare Operations Research Group (ASWORG, later renamed the Operations Evaluation Group (OEG) and now part of the Center for Naval Analyses). The early histories of SRG and ASWORG/OEG were described recently by their original leaders, W.A. Wallis (1980) and P.M. Morse (1977), respectively. While in the SRG, Abraham Wald developed techniques for estimating the survivability of aircraft encountering enemy ground fire. Wald's methods were used in World War II and by the Navy and Air Force during the wars in Korea and Vietnam. Although this work was declassified many years ago, it has not appeared in the open literature. At the end of his historical paper, Wallis (1980) mentions that the Wald work will soon appear in print. The papers Wald wrote describing the methods were recently reprinted by the Center for Naval Analyses (Wald 1980); there are eight memoranda, totaling over 100 pages.

The primary goal of this article is to present an expository survey of Wald's work. Wald's work is interesting from several perspectives. It is of historical interest, since the questions Wald addressed were most urgent at the time but are substantively different from questions of interest to the defense establishment today. Second, Wald's work is interesting in the light of more recent developments (e.g., isotonic regression and numerical methods in missing data problems). It is interesting in a third way, too, for it gives us another example of a great mind in action.

In writing this exposition, we have tried to stay as close to Wald's work as possible. We have followed the logical order of the arguments in the order in which he wrote the memoranda. The work is quite complicated, and many of the details are quite technical. For ease of exposition, we have eliminated as many details as possible while attempting to retain cohesiveness and clarity. The reader interested in full details can obtain copies of the original memoranda from the Center for Naval Analyses.

In the next section, the operational and statistical problems are formulated, some sample data are given, and an overview of the SRG memoranda is given. Section 3 is a survey of Wald's work, beginning with the derivation of his basic equation. Various bounds and approximations for the survivability are then derived. The section concludes with a discussion of the effects of sampling errors. In the last section, we reexamine Wald's work in light of classical statistical theory as well as more recent work. This reexamination leads us to the general conclusion that Wald's treatment of these problems was definitive.

2. THE PROBLEMS AND AN OVERVIEW OF WALD'S WORK

2.1 The Operational and Statistical Problems

The *operational* problem can be stated as follows. Aircraft returning from missions have hits by enemy weapons distributed over various parts of the plane (e.g., wings, tail, fuselage, etc.). The operational commander must decide (a) what tactics would improve survivability, and (b) how to reinforce various parts of the aircraft to improve survivability. Reinforcement means, of course, that the aircraft is heavier, and this impairs its mission. The operational commander does not know the distribution of hits on an aircraft that did not return. This is the basic difficulty in making a decision.

The statistical treatment of the problems that Wald studied is complicated by the fact that data on downed aircraft are unobservable. If these missing data were available, survival probabilities could be estimated by the methods of isotonic regression. Without such data, Wald

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Applications Section

set to work on the problem of estimating the probability that an aircraft that has sustained a fixed number of hits will survive an additional hit. He also attempted to estimate the survival probability of an aircraft sustaining a hit to one of various portions of the body, with different failure rates (e.g., a hit to the nose is more lethal than a hit to the middle of the fuselage). Wald's problems were compounded by a lack of modern computing equipment, a present-day recourse when one is faced with hard problems that resist analytical solution.

2.2 A Hypothetical Set of Data

Throughout the memoranda, Wald used data to illustrate his methods. Although Wald used different data values to illustrate the analysis, we have redone the calculations for one set of data. This helps one see the usefulness of the more complicated analyses.

The set of data is divided into two subsets. The first subset pertains only to hits on the aircraft, ignoring location of the hit. Assume that 400 aircraft were sent on a mission and that the numbers of aircraft returning with i hits anywhere, A_i, are $A_0 = 320, A_1 = 32, A_2 = 20, A_3 = 4, A_4 = 2$, and $A_5 = 2$. The second subset assumes that the location of the hits is known. Subdivide the aircraft into 4 main parts: engines (part 1), fuselage (part 2), fuel system (part 3), everything else (part 4), and let $\gamma(i)$ be the fraction of the area of the aircraft occupied by part i. The total number of hits to all returning aircraft in this case is $\sum_{i=1}^{5} iA_i = 102$. Assume that the hits are distributed according to the following observations:

Part number	$\gamma(i)$	Number of hits (N_i) observed on part
1	.269	19
2	.346	39
3	.154	18
4	.231	26

In anticipation of what follows, let $\delta(i)$ be the fraction of hits observed on part i. Then $\delta(1) = .186$, $\delta(2) = .382$, $\delta(3) = .176$, $\delta(4) = .255$.

These are the kinds of data that the operational commander would obtain and pass on to the statistician working for him. We suggest that the reader now reread the operational problems described in Section 2.1, consider the data again, and then decide how one might attack the problem.

2.3 An Outline of Wald's SRG Memoranda

The basic observational variables are the number N of aircraft participating in the combat, the number A_i of aircraft returning with i hits, and $a_i = A_i/N$. From these data, one wants to find P_i, the probability that an aircraft is downed by the ith hit, and p_i, the conditional probability that an aircraft is downed by the ith hit, given that it received at least $i - 1$ hits and was not downed.

Wald then introduced distributions of hits over the aircraft and found analogous quantities for each subregion of the aircraft. Figure 1 is a flowchart of Wald's work on this problem.

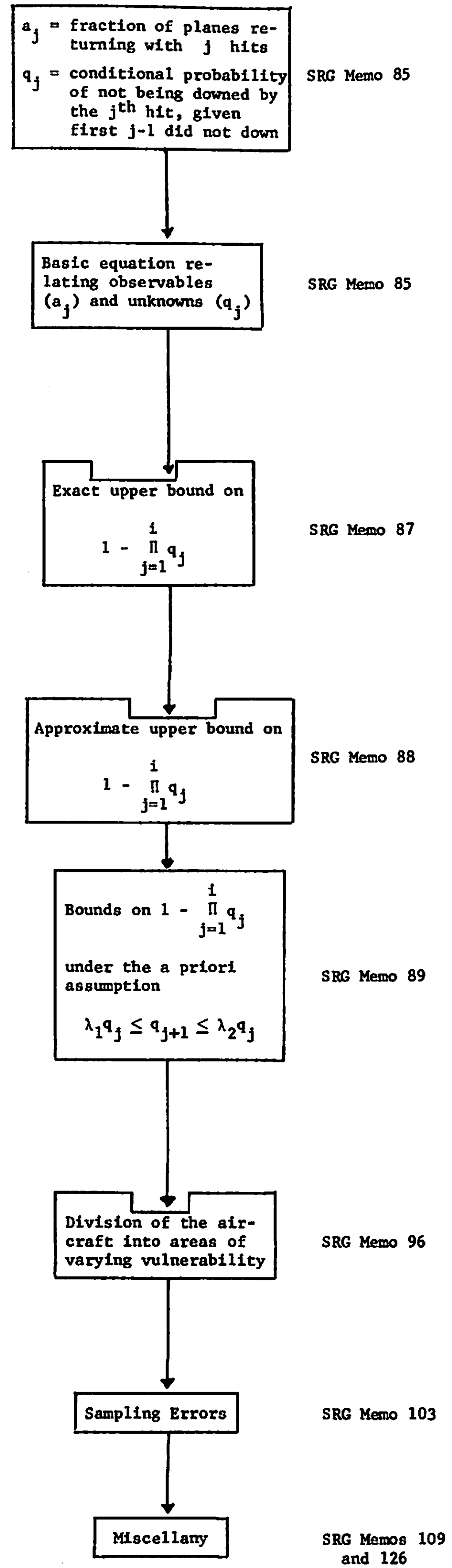

Figure 1. Schematic Outline of Wald's Memoranda.

3. SURVEY OF WALD'S MEMORANDA

This section is a survey of the memoranda. Until Section 3.6, it is assumed that sampling errors are negligible.

3.1 Wald's Basic Equation

In this section, we derive the basic equation satisfied by the probabilities P_i (or $q_i \equiv 1 - p_i$). Let $a_i \equiv A_i/N$ be the fraction of aircraft returning with i hits. Wald assumed that $a_i = 0$ if $i > n$, for some n. Thus, the fraction of aircraft lost is $L = 1 - \sum_{i=0}^{n} a_i$. Wald also assumed that an unhit aircraft always returns and that there is a value m such that the probability of receiving more than m hits is zero. He argued that $m = n + 1$.

Let x_i be the fraction of aircraft downed by the ith hit. (Thus $x_0 \equiv 0$.) Then $\sum_{i=0}^{n} x_i = L$. The x_i's then satisfy the recursion relationship

$$x_i = p_i\left(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=0}^{i-1} x_j\right), \quad i = 1, \ldots, n. \tag{3.1}$$

The term in brackets in (3.1) is the proportion of aircraft receiving at least i hits. If c_i is defined by $c_i = 1 - \sum_{j=0}^{i-1} a_j$, then (3.1) becomes

$$x_i + p_i \sum_{j=0}^{i-1} x_j = p_i c_i, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

For some of the analysis, Wald found (3.2) more useful than (3.1). The goal now is to somehow relate the observables (a_j) to the probabilities. In SRG 85, Wald derives the following equation, which is basic to all of his work.

$$\sum_{j=1}^{n} \frac{a_j}{q_1 \cdots q_j} = 1 - a_0. \tag{3.3}$$

Equation (3.3) relates the observables a_j, the fractions of aircraft returning with j hits, and the unknowns q_j, the conditional probability of not being downed by the jth hit given that the first $j - 1$ hits did not down the aircraft. It is the fundamental equation of the analysis. In the next section, we compare Wald's work with other approaches to this problem. For this reason, it helps to review Wald's derivation of (3.3).

Let b_i be the hypothetical proportion of aircraft hit i times if dummy bullets were used. Then $b_i \geq a_i$; set $y_i = b_i - a_i$. In addition, $y_i = P_i b_i = P_i(a_i + y_i)$. Thus $y_i = (P_i/Q_i)\, a_i$, where as before, $P_i = 1 - q_1 q_2 \cdots q_i$ and $Q_i = q_1 \cdots q_i$. Hence we obtain $y_i = (a_i/q_1 \cdots q_i) - a_i$. Summing up to n and noting that $\sum_{i=1}^{n} y_i = L$ gives (3.3).

Equation (3.3) is easily solved with the simplifying assumption of constant $q_j \equiv q$. For example, for the data, (3.3) becomes the fifth-order equation

$$\frac{.08}{q} + \frac{.05}{q^2} + \frac{.01}{q^3} + \frac{.005}{q^4} + \frac{.005}{q^5} = .20, \tag{3.4}$$

which yields $q = .851$. Hence p_i, the probability of the ith hit downing the aircraft given that the first $i - 1$ hits did not down it, is $p_i = .149$ (for all i).

Once we know p_i, we can compute x_i, the ratio of the number of aircraft downed by the ith to the total number of aircraft participating, recursively from Equations (3.1) or (3.2). We find that $x_1 = .02980$, $x_2 = .01344$, $x_3 = .00399$, $x_4 = .00190$, and $x_5 = .00087$.

These results are easily obtained, but are based on the assumption of $q_1 = q_2 = \cdots = q_n$. This severely limits their usefulness. The rest of Wald's memoranda studies ways of relaxing this assumption.

3.2 A Least Upper Bound for the Probability of i Hits Downing an Aircraft

Wald's next step was to find a bound on $P_i = 1 - \prod_{j=1}^{i} q_i$, which is the probability of an aircraft being downed by i hits. The bound he found is sharp and its attainment corresponds to the worst case in terms of survivability.

The problem of interest may be written as follows:

$$\text{minimize} \quad \prod_{j=1}^{i} q_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} \frac{a_j}{q_1 \cdots q_j} = 1 - a_0. \tag{3.5}$$

Equation (3.5) is a nonlinear optimization problem (Avriel 1976). Wald obtained an iterative solution as follows. First he showed that if a set $\{q_1^*, \ldots, q_n^*\}$ solves (3.5), then $q_i^* = q_{i+1}^* = \cdots = q_n^*$; that is, that the q_j are all equal for $j \geq i$.

Applying this result when $i = 1$ means that q_1 is minimized if it satisfies

$$\sum_{j=1}^{n} \frac{a_j}{q_1^j} = 1 - a_0. \tag{3.6}$$

Assume now that q_1 is known by solving the algebraic equation (3.6). Next one needs to find the value of q_2 that minimizes $q_1 q_2$. Using the result given above, problem (3.5) becomes

$$\text{minimize} \quad q_1 q_2$$

$$\text{subject to} \quad \frac{1}{q_1} \sum_{j=1}^{n} \frac{a_j}{q_2^{j-1}} = 1 - a_0. \tag{3.7}$$

Straightforward solution via the Lagrange multiplier method gives

$$q_1 = \frac{1}{1 - a_0} \sum_{j=2}^{n} \frac{(j-1)a_j}{q_2^{j-1}}$$

and

$$\sum_{j=2}^{n-1} \frac{(j-1)a_{j+1}}{q_2^j} = a_1. \tag{3.8}$$

Elementary arguments show that these equations have exactly one root in (q_1, q_2).

Wald then generalized this argument to determine the minimum of $\prod_{j=1}^{i} q_j$. He followed the same kind of reasoning, starting with the assumption that $q_j = q_2$, $i \geq j \geq 2$; then one wants to minimize $q_1 q_2^{i-1}$. The Lagrange multiplier method is used again; only the details change.

It is clear that even with present-day computing abilities this approach quickly becomes complicated and time-consuming. In 1943, the task of exact computations was hopeless for any problems of operational interest; thus Wald considered various approximation schemes.

3.3 Approximate Bounds on P_i

Wald's next step was to obtain approximate upper and lower bounds on P_i. Let P_i^* be the maximum value of P_i and let $Q_i^* = 1 - P_i^*$. The first step is to find the lower bound z_i of Q_i^*, that is, to find a bound on the minimum of Q_i. Wald used an interesting kind of hypothetical reasoning: Let y_j be the fraction of the returning aircraft that would be downed if they were to receive $i - j$ additional hits. Then one obtains

$$P_i = \sum_{j=0}^{i-1} y_j + \sum_{j=1}^{i} x_j, \quad i = 1, 2, \ldots, n. \tag{3.9}$$

After some algebraic manipulations, Wald obtained the bounds

$$1 - \frac{\sum_{j=1}^{i} x_j}{\left(1 - \sum_{j=0}^{i-1} a_j\right)} < Q_i < 1 - \sum_{j=1}^{i} x_j. \tag{3.10}$$

Equation (3.10) provides a lower bound on Q_i, once an upper bound on $\sum_{j=1}^{i} x_j$ is known. Wald showed that the maximum value of $X_i \equiv \sum_{j=1}^{i} x_j$ occurs when $p_1 = p_2 = \cdots = p_n = p$. In such a case, the solution of (3.6) gives $q_1 = 1 - p$, and then the x_i are obtained from (3.1). We will let z_i be the lower bound on Q_i obtained in this manner.

Next Wald turned to the problem of estimating an upper bound on the value of Q_i. He showed that such an upper bound is given by

$$t_i = \min[\bar{u}_1^{\,i}, \bar{u}_2^{\,i-1}, \ldots, \bar{u}_{i-1}^{\,2}, \bar{u}_i], \tag{3.11}$$

where $\bar{u}_r$ is the positive root of the equation

$$\sum_{j=r}^{n} \frac{a_j}{u^{j-r+1}} = 1 - \sum_{j=0}^{r-1} a_0. \tag{3.12}$$

He obtained (3.11) and (3.12) by a sequence of manipulations on equations analogous to (3.5) and (3.6).

Let us now apply these results to the data. First we will find the lower bound z_i. The first step is to find q_0, the solution of (3.3) when $q_1 = q_2 = \cdots = q_n$. In this case, we have found q_0 as the solution of (3.4); that is, $q_0 = .851$. We have also found the x_i and thus obtain the upper bounds $X_i = \sum_{j=1}^{i} x_j$. For the data, we obtain $X_1 = .02980$, $X_2 = .04324$, $X_3 = .04723$, $X_4 = .04913$, $X_5 = .05000$. According to (3.10), our lower bound is $z_i = 1 - (X_i/(1 - \sum_{j=0}^{i-1} a_j))$. Hence we obtain $z_1 = .85100$, $z_2 = .63967$, $z_3 = .32529$, and $z_4 = .18117$. It is not necessary to calculate z_5, since Q_5 can be obtained directly. In this case, $z_5 = .090909$.

Now consider the upper bounds t_i. Let us write out some of the Equations (3.12), to see what they look like. For $r = 1$, we obtain (3.4), so that $\bar{u}_1 = .851$. For $r = 2$, (3.12) becomes

$$\frac{a_2}{u} + \frac{a_3}{u^2} + \frac{a_4}{u^3} + \frac{a_5}{u^4} = 1 - a_0 - a_1, \tag{3.13}$$

which has solution $\bar{u}_2 = .722$. In a similar way, one finds $\bar{u}_3 = .531$, $\bar{u}_4 = .333$. The t_i are given by (3.11); namely

$$\begin{aligned} t_1 &= .851 \\ t_2 &= \min(\bar{u}_1^{2}, \bar{u}_2) = .722 \\ t_3 &= \min(\bar{u}_1^{3}, \bar{u}_2^{2}, \bar{u}_3) = .521 \\ t_4 &= \min(\bar{u}_1^{4}, \bar{u}_2^{3}, \bar{u}_3^{2}, \bar{u}_4) = .282. \end{aligned} \tag{3.14}$$

Note that t_5 is not calculated since the exact value of Q_5 can be found.

In Table 1, we compare the exact result obtained by the method of the previous section with lower bound (z_i), upper bound (t_i), and the value obtained assuming all hits are equally lethal ($q_0^{\,i}$).

3.4 Bounds on P_i Under Additional Assumptions

The results of the previous section are, from a computational viewpoint, less cumbersome than the exact results. They are still complicated to use, however, so Wald studied the bounds on survival probability under additional assumptions. These assumptions are that

$$\lambda_1 q_j \leq q_{j+1} \leq \lambda_2 q_j, \quad j = 1, 2, \ldots, n - 1 \tag{3.15}$$

for fixed known λ_1 and λ_2, and that

$$\sum_{j=1}^{n} a_j \lambda_1^{-j(j-1)/2} < 1 - a_0. \tag{3.16}$$

Note that (3.16) need not be true if λ_1 is too small; but if λ_1 is close enough to 1, then (3.16) will be true. The basic Equations (3.3) and (3.16) imply that $q_1 < 1$.

Wald first calculated the values of $q_1, \ldots, q_n$, which make Q_i ($i < n$) a minimum. Denote these by $q_1^*, \ldots, q_n^*$. By using a straightforward proof by contradiction, he proved the following: (a) for $j = i, i + 1, \ldots, n - 1$, $q_{j+1}^* = \lambda_2 q_j^*$; and (b) if j is the smallest integer such

Table 1. Exact and Approximate Values of Q_i

i	Value: Exact Value	Lower Bound	Upper Bound	Equal Lethality of Hits
1	.851	.851	.851	.851
2	.721	.640	.722	.724
3	.517	.325	.521	.616
4	.282	.181	.282	.525

that $q_{k+1}^* = \lambda_2 q_k^*$ for all $k \geq j$, then $q_r^* = \lambda_1 q_{r-1}^*$ for $r = 2, 3, \ldots, j - 1$. These results can be viewed as analogs of the results in Section 3.3.

Let E_{ir}, $r = 1, \ldots, i - 1$, be the minimum value of Q_i under the restriction that $q_{j+1} = \lambda_1 q_j$, $j = 1, \ldots, r - 1$, and $q_{j+1} = \lambda_2 q_j$ for $j = r + 1, \ldots, n - 1$. The above results show that $Q_i = \min\{E_{i1}, E_{i2}, \ldots, E_{i,i-1}\}$. The results in Sections 3.2 and 3.3 show how the E_{ir} can be calculated. In particular, Wald showed that if g_r is the positive root (in q) of the equation (for $r = 0, 1, 2, \ldots, i - 1$)

$$\sum_{j=1}^{r+1} a_j \lambda_1^{-j(j-1)/2} q^{-j} + \sum_{j=1}^{n-r-1} \{a_{r+1+j} \lambda_1^{-r(r+1)/2 - rj}\} \times \{\lambda_2^{-j(j+1)/2} q^{-(r+1+j)}\} = 1 - a_0 \quad (3.17)$$

then an approximation to E_{ir} is

$$E_{ir} \simeq \lambda_1^{r(r+1)/2 + r(i-r-1)} \lambda_2^{(i-r)(i-r-1)/2} q_r^i. \quad (3.18)$$

Similar arguments show that if $q_1^*, \ldots, q_n^*$ are values of q_j minimizing $Q_n = \prod_{j=1}^n q_j$, then $q_{j+1}^* = \lambda_1 q_j^*$, $j = 1, \ldots, n - 1$. This means that if q is the root of the equation

$$\sum_{j=1}^{n} a_j \lambda_1^{-j(j-1)/2} q^{-j} = 1 - a_0, \quad (3.19)$$

then the minimum value of Q_n is $\lambda_1^{n(n-1)/2} q^n$.

Wald proceeded in the same fashion to show that the maximum of Q_n is $\lambda_2^{n(n-1)/2} q^n$, where q is a solution of the (3.19) with λ_1 replaced by λ_2.

There is a quantity analogous to E_{ir}. Namely, if D_{ir} is the maximum of Q_i under the restriction that $q_{j+1} = \lambda_1 q_j$ for $j = r + 1, \ldots, n - 1$ and $q_{j+1} = \lambda_2 q_j$ for $j = 1, \ldots, r - 1$, then Wald showed that the maximum of Q_i is $\max\{D_{i1}, \ldots, D_{i,i-1}\}$. He showed that a good approximation to D_{ir} is obtained from (3.17) and (3.18) with the λ_1 and λ_2 interchanged.

We apply these results to the data with $\lambda_1 = .85$, $\lambda_2 = .95$. It is easy to check that (3.16) is satisfied.

To find the lower limit of Q_i, the four equations (for $r = 0, 1, 2, 3$) (3.17) must be solved. For example, for $r = 0$ this equation is

$$\frac{a_1}{q} + \frac{a_2}{\lambda_2 q^2} + \frac{a_3}{\lambda_2^3 q^3} + \frac{a_4}{\lambda_2^6 q^4} + \frac{a_5}{\lambda_2^{10} q^5} = 1 - a_0. \quad (3.20)$$

The roots of (3.17) for the values $r = 0, 1, 2, 3$ are $g_0 = .887$, $g_1 = .938$, $g_2 = .964$, and $g_3 = .979$. Next, the E_{ir} are found approximately from (3.18), and then Q_i is the minimum of the E_{ir}. Table 2 shows the results of such calculations. The lower limit of Q_5 is found by using (3.19). In this case, the root of (3.19) is $q = .986$ and the lower limit of $Q_5 = \lambda_1^{10} q^5$ is .183.

To find the maximum value of Q_i, the same procedure is followed. Since the details are the same, only the final results will be given. Table 3 shows both bounds.

Table 2. Estimating the Minimum of the Survival Probability

i	r	g_r	E_{ir} Approximately	min Q_i Approximately
1	0	.887	.887	.887
2	0	.887	.747	
	1	.938	.747	.747
3	0	.887	.598	
	1	.938	.567	
	2	.964	.550	.550
4	0	.887	.455	
	1	.938	.408	
	2	.964	.364	
	3	.979	.347	.347

3.5 Analysis of Vulnerability Areas of the Aircraft

Wald considered next the problem of determining the vulnerability of different parts of the aircraft. The idea here is that the location of the hits on returning aircraft provides useful information on the vulnerability of various parts of the aircraft. Wald began with the premise that one knows the conditional probability $\gamma_i(i_1, \ldots, i_k)$ that area m will receive i_m hits given a total of $i = \sum_{m=1}^k i_m$ hits. He argued that $\gamma_i(i_1, \ldots, i_k)$ can be experimentally determined by firing dummy bullets at real aircraft. The quantity of interest here is $Q_i(i_1, \ldots, i_k)$, the probability that an aircraft is not downed given i_m hits to area m, with $\sum_{m=1}^k i_m = i$. Wald first formulated the problem in a very general setting, where it is essentially intractable.

To make any progress, he needed to introduce an assumption of independence. Thus, he assumed that if $q(i)$ is the probability that one hit on area i will not down the aircraft and if $\gamma(i)$ is the conditional probability that area i is hit given that one hit occurred, then

$$Q_i(i_1, \ldots, i_k) = \prod_{m=1}^k [q(m)]^{i_m}, \quad (3.21)$$

$$\gamma_i(i_1, \ldots, i_k) = \frac{i!}{\prod_{m=1}^k i_m!} \prod_{m=1}^k [\gamma(m)]^{i_m}. \quad (3.22)$$

In (3.21) and (3.22), it is understood that $\sum_{m=1}^k i_m = i$. Let $\delta(i)$ be the probability that area i is hit, given that the aircraft received exactly one hit that did not down it. Then

Table 3. Lower and Upper Bounds on Q_i

i	Lower Bound on Q_i	Upper Bound on Q_i
1	.887	.986
2	.747	.826
3	.550	.631
4	.347	.463
5	.183	.329

by its definition

$$\delta(i) = \frac{\gamma(i)q(i)}{\sum_{i=1}^{k} \gamma(i)q(i)}. \tag{3.23}$$

In (3.23), recognize the summation as the probability q that a single shot did not down the aircraft. Under the assumption of independence, q will satisfy (3.3) with $q_j \equiv q$ and may be replaced by the solution to that equation. Equation (3.23) is rewritten as

$$q(i) = \frac{\delta(i)q}{\gamma(i)}, \tag{3.24}$$

where $\gamma(i)$ is assumed to be known from auxiliary tests or equated with the proportion of surface area associated with part i, and $\delta(i)$ may be estimated from the data as

$$\delta(i) = \frac{\sum_{j_k} \cdots \sum_{j_1} j_i a(j_1, \ldots, j_k)}{\sum_{j_k} \cdots \sum_{j_1} (j_1 + \cdots + j_k) a(j_1, \ldots, j_k)}. \tag{3.25}$$

The interpretation of $\delta(i)$ is that it is the ratio of the total number of hits in area i of the returning aircraft to the total number of hits on the returning aircraft. Thus, $\delta(i)$ is empirically determined and $q(i)$ is computed by applying (3.23) to the data. Such analyses have actually been performed on real data, with success.

We apply this approach to the data. We have already seen that the positive root of (3.3) with equal q_j is $q_0 = .851$. Thus q_0 is the overall probability of surviving a hit. The probability of surviving a hit to part i is given by (3.24). The q in (3.4) is q_0; $\gamma(i)$ (the fraction of area occupied by part i) and $\delta(i)$ (the fraction of hits to part i) were given along with the data. The results of the calculations are shown in Table 4. For these data, the most vulnerable portion of the aircraft is the engine area.

3.6 Effects of Sampling Errors

Wald considered sampling errors in the special case of equal (but unknown) q_j, and he derived confidence limits for q.

In the absence of sampling errors, the x_i are recursively defined by (3.1) with equal p_i. When there are sampling errors, (3.1) is replaced by

$$x_i = \bar{p}_i\left(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i-1} x_j\right), \tag{3.26}$$

Table 4. Probability of Surviving a Single Hit to a Given Part

Part	*Probability of Surviving a Single Hit*
Entire Aircraft	.851
Engine	.588
Fuselage	.940
Fuel System	.973
Others	.939

where $\bar{p}_i$ has the distribution of the success ratio in a sequence of $N_i = N(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i-1} x_j)$ independent trials. Still assuming that $x_i = 0$ for $i > n$ (which is not really true for the case with sampling errors), the basic equation (3.3) becomes

$$\sum_{j=1}^{n} \frac{a_j}{\bar{q}_i \cdots \bar{q}_j} = 1 - a_0. \tag{3.27}$$

Here $\bar{q}_j = 1 - \bar{p}_j$ is an estimate for q; but the $\bar{q}_j$'s are unknown.

Wald derived confidence bounds in the following manner. Consider a hypothetical experiment in which b_i is the fraction of aircraft that would be hit exactly i times if dummy bullets were used. The distribution of Na_i is the same as the distribution of the number of successes in a sequence of Nb_i independent trials, each trial having a probability of success q^i. This gives

$$E(a_i/q^i) = b_i, \quad \operatorname{var}(a_i/q^i) = \frac{b_i(1 - q^i)}{Nq^i}. \tag{3.28}$$

Summing (3.28) gives

$$E\left(\sum_{i=1}^{n} a_i/q^i\right) = \sum_{i=1}^{n} b_i = 1 - a_0,$$

$$\operatorname{var}\left(\sum_{i=1}^{n} \frac{a_i}{q^i}\right) = \sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}. \tag{3.29}$$

For moderate to large N, appeal to the central limit theorem and conclude that if

$$\int_{-\lambda_\alpha}^{\lambda_\alpha} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \alpha,$$

then an α confidence interval for q is

$$1 - a_0 - \lambda_\alpha\left(\sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}\right)^{1/2} \le \sum_{i=1}^{n} \frac{a_i}{q^i} \le 1 - a_0 + \lambda_\alpha\left(\sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}\right)^{1/2}. \tag{3.30}$$

The only trouble with (3.30) is that the b_i are not known. Again appealing to limit theorems, Wald replaced b_i by a_i/q^i (this replacement is accurate to $O(1/\sqrt{n})$). Hence we obtain a confidence interval of the form

$$1 - a_0 - \lambda_\alpha\left(\sum_{i=1}^{n} \frac{a_i(1 - q^i)}{Nq^{2i}}\right)^{1/2} \le \sum_{i=1}^{n} \frac{a_i}{q^i} \le 1 - a_0 + \lambda_\alpha\left(\sum_{i=1}^{n} \frac{a_i(1 - q^i)}{Nq^{2i}}\right)^{1/2}. \tag{3.31}$$

A final simplification is achieved by another appeal to a limit theorem. If q_0 is the root of (3.3) with equal q_i, then as $N \to \infty$, $q \to q_0$, so Wald replaced q^{2i} by q_0^{2i} in (3.31), and the resulting confidence limit is now very simple.

These results can be summarized in the following elegant fashion. If a_i are subject to sampling error and q is the true parameter, then $\sum_{i=1}^{n} a_i/q^i$ is normally distributed with mean $1 - a_0$ and variance given by (3.29).

To show how this works, we will derive the 95% and 99% confidence intervals for the data. The first step is to find the positive solution, q_0, of (3.3) with equal q_j. In this case, $q_0 = .851$. The second step is to find the approximate variance of $\sum_{i=1}^{n} a_i/q^i$. This variance is

$$\sigma^2 = \sum_{i=1}^{n} a_i(1 - q_0^i)/Nq_0^{2i}, \tag{3.32}$$

and in this case we find $\sigma = .01373$. According to (3.31), the confidence limits are found by solving

$$\sum_{i=1}^{n} \frac{a_i}{q^i} = 1 - a_0 \pm \lambda_\alpha \sigma, \tag{3.33}$$

where $\lambda_\alpha = 1.960, 2.576$ for the 95% and 99% limits, respectively. For the 95% confidence limit on q_0, the solution of (3.33) gives [.797, .921] and for the 99% confidence limit, [.782, .947].

3.7 Miscellany

SRG Memoranda 109 and 126 deal, very briefly, with these topics: (a) factors that are nonconstant in combat, (b) nonprobabilistic interpretation of the results, (c) the situation when $\gamma(i)$ are unknown, and (d) vulnerability to different kinds of guns. The most interesting of these topics is the last one, in which Wald generalizes the previous work to include different kinds of weapons. Namely, instead of working with $q(i)$, the probability that an aircraft survives a hit to part i, he works with $q(i, j)$, the probability that an aircraft survives a hit to part i by weapon type j. The generalization is conceptually straightforward, although the details are complicated.

4. DISCUSSION

In this section, we propose to reexamine Wald's work on aircraft survivability, relating his results to classical statistical theory as well as to more recent statistical thought. We believe that such a development makes Wald's recommendations more easily understood. It also allows us to support the general conclusion that Wald's treatment of this problem was definitive, since, through this reexamination, we are able to identify the optimal character of Wald's estimators and to explain why treatment of more general problems is impossible with the data Wald had available to him.

Let us consider the first data set. Wald does not explicitly discuss a model for the data he seeks to fit. It is clear, however, that the appropriate model is multinomial. It is also clear that there are missing data. It is useful to picture the data as embedded in the following scheme.

$$\begin{array}{cccccc} X_{01} & X_{11} & X_{21} & X_{31} & X_{41} & X_{51} \\ & X_{12} & X_{22} & X_{32} & X_{42} & X_{52} \end{array} \tag{4.1}$$

where X_{i1} = the number of aircraft returning with i hits, and X_{i2} = the number of aircraft downed with i hits. Data Set 1 amounts to X_{i1}, $i = 0, \ldots, 5$, while X_{i2}, $i = 1, \ldots, 5$ are unobservable. The multinomial distribution based on 400 observations classified into 11 cells represents the full model for the collection $\{X_{ij}\}$. Let the parameters of the full model be denoted by $\{p_{ij}\}$. Wald prefers to use the parameterization:

(1) $p_{01}, \ldots, p_{51}$ (for which $a_0, \ldots, a_5$ are the corresponding sample proportions in Wald's notation)

(2) $Q_1, \ldots, Q_5$, where

$$Q_i = \frac{p_{i1}}{p_{i1} + p_{i2}}. \tag{4.2}$$

Whatever the parameterization, the critical fact vis-a-vis the estimation problem of interest is that the full model is determined by 10 parameters while the available data have only six degrees of freedom. Put another way, the 10-parameter model for the available data is not identifiable; indeed, the likelihood depends on $\{p_{12}, \ldots, p_{52}\}$ only through the value of $\sum_{i=1}^{5} p_{i2}$. The nonidentifiability of the model for X_{i1}, $i = 0, \ldots, 5$ explains the role of the assumption

$$Q_i = q^i \quad \text{for all } i. \tag{4.3}$$

This restriction renders the estimation problem well defined. The necessity of identifiability also dictates the assumption (for the purpose of analyzing the data set) that the probability of sustaining more than five hits is zero.

We now turn to the derivation of the maximum likelihood estimators for the parameters of the multinomial distribution with missing data under the restriction (4.3). Initially, we write the likelihood as

$$\mathcal{L} \propto \left(\prod_{i=0}^{5} p_{i1}^{x_{i1}}\right)\left(1 - \sum_{i=0}^{5} p_{i1}\right)^{400 - \sum_{i=0}^{5} x_{i1}}.$$

The likelihood equations

$$\left\{\frac{\partial}{\partial p_{i1}} \mathcal{L} = 0\right\}_{i=0}^{5}$$

are equivalent to

$$\hat{p}_{i1} = \frac{x_{i1}}{N} \qquad i = 0, \ldots, 5.$$

Now, the parametric analog of Wald's fundamental equation (3.3) is

$$\sum_{j=1}^{n} \frac{p_{j1}}{\prod_{i=1}^{j} q_i} = 1 - p_{01}. \tag{4.4}$$

The latter equation can be shown to be algebraically equivalent to

$$\sum_{j=1}^{n} (p_{j1} + p_{j2}) = 1 - p_{01}, \tag{4.5}$$

which simply specifies that all cell probabilities sum to one. Under restriction (4.3), Equation (4.4) becomes

$$\sum_{j=1}^{n} \frac{p_{j1}}{q^j} = 1 - p_{01}, \qquad (4.6)$$

specifying q implicitly as a function of $\{p_{i1}, i = 0, \ldots, n\}$. Now, let $\hat{q}$ be the solution of (3.3), which, for the first data set, can be written as

$$\sum_{j=1}^{5} \frac{\hat{p}_{j1}}{\hat{q}^j} = 1 - \hat{p}_{01}. \qquad (4.7)$$

From the invariance property of the MLE's, it is clear that $\hat{q}$ is the MLE of the parameter q.

The regularity of the multinomial model implies the asymptotic optimality of Wald's estimators of the parameters $\{p_{i1}\}$ and p. Wald's confidence interval for the survival probability q can be obtained via MLE theory and thus, its optimality in large samples can be asserted. Since interesting larger models cannot be treated with the data available, Wald's estimation results are, with a sufficiently large sample size, the best possible. For larger models, Wald appropriately turns to the development of bounds on survival probabilities.

Two important areas of statistical analysis having some bearing on Wald's work have been developed since Wald's time. The first is the area of isotonic regression, a subject treated in depth in the recent book by Barlow et al. (1972). The second is the treatment of problems with missing data via the EM algorithm (see Dempster, Laird, and Rubin 1977). Isotonic regression would appear to be an appropriate methodology in Wald's problem, since aircraft vulnerability undoubtedly increases with the number of hits sustained; that is, it is reasonable to expect that $p_1 \leq p_2 \leq \cdots \leq p_n$. In spite of its intuitive appeal, the isotonic version of Wald's problem suffers from nonidentifiability, since ordering of parameters does not reduce the dimension of the parameter space. Thus, given Wald's data, estimation via the methods of isotonic regression proves impossible without additional assumptions. If complete data were available, the unrestricted MLE's for the q_i's are given by

$$\prod_{j=1}^{i} \hat{q}_j = \frac{x_{i1}}{x_{i1} + x_{i2}} \quad i = 1, \ldots, 5. \qquad (4.8)$$

The problem of "isotonizing" these estimates is formally equivalent to the problem of estimating ordered binomial parameters treated by Barlow et al. (1972, p. 102).

The EM algorithm does not help for similar reasons. When the model is not identifiable, a starting value $\mathbf{p}^{(0)}$ for the parameter produces expected $\mathbf{X}$ values, which in turn produce $\mathbf{p}^{(1)} = \mathbf{p}^{(0)}$. In the reduced model, subject to (4.3), one can treat maximum likelihood estimation analytically, and there is no need to employ the EM algorithm.

Let us now examine Wald's estimators for the survival probabilities of various aircraft sections. The portion of the data set classifying hits by part can be viewed as embedded in the array

$$\begin{array}{ccccc} Y_{11} & Y_{21} & Y_{31} & Y_{41} & N_1 \\ Y_{12} & Y_{22} & Y_{32} & Y_{42} & N_2 \end{array} \qquad (4.9)$$

where Y_{i1} = # of hits to part i on returning aircraft; Y_{i2} = # of hits to part i on downed aircraft; $N_1 = \sum_{i=1}^{4} Y_{i1}$; $N_2 = \sum_{i=1}^{4} Y_{i2}$. The data consist of Y_{i1}, $i = 1, \ldots, 4$ and N_1, while Y_{i2}, $i = 1, \ldots, 4$ and N_2 are unobservable. Define the following events:

$$A_i = \{\text{the } i\text{th section is hit}\}$$

$$A = \{\text{the aircraft is hit}\}$$

$$B = \{\text{the aircraft is not downed}\}.$$

Wald's parameters may be identified as

$$q = P(B \mid A),\ q(i) = P(B \mid A_i)$$

$$\delta(i) = P(A_i \mid A \cap B),\ \gamma(i) = P(A_i \mid A). \qquad (4.10)$$

With complete data as pictured in (4.9), the MLE's of $q(i)$ are simply

$$\hat{q}(i) = \frac{Y_{i1}}{Y_{i1} + Y_{i2}} \quad i = 1, \ldots, 4. \qquad (4.11)$$

With the incomplete data available to Wald, one must make use of the structural relationship (3.23) (which is immediate from the definitions in (4.10)) and the assumption that $\gamma(i)$, $i = 1, \ldots, 4$ are known. Wald explicitly remarks on the impossibility of estimating $\gamma(i)$ and $q(i)$ simultaneously from his data. However, MLE's for $\{\delta(i)\}$ and q may be obtained from the data, and the estimates

$$\hat{q}(i) = \frac{\hat{\delta}(i)}{\gamma(i)} \cdot \hat{q} \quad i = 1, \ldots, 4 \qquad (4.12)$$

are maximum likelihood estimates by invariance, provided these estimates lie in the unit interval. Wald does not deal with estimation problems in which one or more of the estimates $\hat{q}(i)$ exceed one. In such cases, the MLE of the vector $(q(1), \ldots, q(4))$ lies on the boundary of the parameter space, and its identification is tedious but straightforward.

In our discussion of Wald's formulation and solution of a variety of problems dealing with aircraft survivability, we have mentioned a number of assumptions he imposed to obtain closed-form solutions or efficient bounds. These assumptions deserve scrutiny. Among the assumptions one encounters are (a) constant vulnerability, that is, $q_i \equiv q$, which is an independence assumption; (b) known bounds on rate of growth of vulnerability, that is, $\lambda_1 q_j \leq q_{j+1} \leq \lambda_2 q_j$; and (c) independence of survival among and within areas of different vulnerability. The main cause for concern regarding these assumptions is that the data available do not provide a means for investigating their validity. Consider assumption (a), for example. With complete data (corresponding to $\{x_{ij}\}$ in (4.1)) one could investigate statistically, via a likelihood ratio test or otherwise, the validity of the assumption $q_i \equiv q$. With the type of data available to Wald, such an option is not open because of the lack of identifiability of larger models. Wald cautioned his readers that the solution he provides should be used only "if it is known *a priori* that $q_1 = q_2 = \cdots = q_n$." How and whether such a priori knowledge could be garnered is open to debate. Wald does provide an option for those who are more conservative. The lower bounds for Q_i may be considered conservative estimates of survival probabilities, although they might often be too small to be useful. The dilemma one encounters with the foregoing three assumptions mentioned is similar to that faced in competing risks methodology, where considerable recent work has focused on identifiability and bounds for survival probabilities (see Tsiatis 1975 and Peterson 1976).

Viewing Wald's work on aircraft survivability in light of the state of the art at the time it was done, it seems to us to be a remarkable piece of work. While the field of statistics has grown considerably since the early 1940's, Wald's work on this problem is difficult to improve upon. Much of the work appears to be ad hoc—there are few allusions to modeling and no reference to classical statistical approaches or results. By the sheer power of his intuition, Wald was led to subtle structural relationships (e.g., Equations (3.3) and (3.24)), and was able to deal with both structural and inferential questions in a definitive way.

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Comment

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The authors are to be congratulated on a fine paper. They have distilled the key ideas in Wald's work on aircraft survivability, and have successfully related the ideas to standard statistical methods. The bulk of this discussion will be concerned with this relationship of the work to standard statistical methods, particularly the use of statistical models to describe the situation. Some attention will also be given to decision-theoretic issues.

1. STATISTICAL MODELING

As indicated in the paper, the primary quantities studied can be considered

$$P_{i1} = P\ (i \text{ hits and survival}) = Q_i \cdot \lambda_i,$$

where

$$Q_i = P\ (\text{survival} \mid i \text{ hits}),$$

$$\lambda_i = P\ (i \text{ hits}),$$

and

$$P_0^* = P\ (\text{not surviving}) = 1 - \sum_{i=0}^{\infty} P_{i1}.$$

If the observations can be assumed to be independent, and out of a total of n missions the data are

X_{i1} = the number of aircraft that receive i hits and survive,

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$$X_0^* = n - \sum_{i=0}^{\infty} X_{i1} = \text{the number that do not survive},$$

then the likelihood function for $\mathbf{P} = (P_0^*, P_{11}, P_{21}, \ldots)$ is proportional to

$$L(\mathbf{P}) = \left(\prod_{i=0}^{\infty} P_{i1}^{X_{i1}}\right) (P_0^*)^{X_0^*}$$

$$= \left[\prod_{i=0}^{\infty} (Q_i \cdot \lambda_i)^{X_{i1}}\right]\left[1 - \sum_{i=0}^{\infty} Q_i \cdot \lambda_i\right]^{X_0^*}. \quad (1)$$

In this framework, which is more or less that given in Section 3 of Mangel and Samaniego, Wald's *model* can be described by the following assumptions:

(i) $Q_i = q^i$ (i.e., iid survival of each hit);
(ii) $P_{i1} = 0$ for $i \geq 6$ (or, more generally, for i for which $X_{i1} = 0$).

We will return to the crucial assumption (i) later, but for now will accept it. Assumption (ii) leaves an obvious uncomfortable feeling, but probably makes no great difference for the type of data expected. A third assumption, actually a lack of an assumption, is also a possible cause for concern: Wald effectively leaves the λ_i (the probability of i hits) completely unrestricted, whereas it would seem more natural to restrict the parameter space to consist only of decreasing λ_i. (Actually, the λ_i are never even mentioned in Wald's work, an omission of some concern, as we shall see.)

As mentioned in the paper, Wald's analysis effectively corresponds to a maximum likelihood analysis using (1) and assumptions (i) and (ii). The results of this analysis for the given data are $\hat{q} = .851$ and $\hat{\lambda}_i = X_{i1}/[400\,(.851)^i]$. The values of the $\hat{\lambda}_i$ for the data are given in Table 1, and indeed they are not decreasing ($\hat{\lambda}_5 > \hat{\lambda}_4$). The possible difference here seems minor but, as a theoretical point, it seems desirable to ensure monotonicity of the λ_i in the analysis. (Perhaps the most straightforward way of incorporating monotonicity is simply to put the (noninformative) uniform prior distribution on

$$\Lambda = [(\lambda_0, \ldots, \lambda_5)\colon \textstyle\sum \lambda_i = 1, \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_5\},$$

a uniform prior on q (in [0,1]), and calculate the posterior means, providing the numerical integration problem is feasible.)

The most significant question that can be raised concerning Wald's analysis is that of overparameterization. The parameters are $(q, \lambda_0, \ldots, \lambda_5)$, seven parameters for the seven data values $(X_0^*, X_{01}, \ldots, X_{51})$. Wald attempts a model robustness study by finding lower and upper bounds for the P_{i1} (actually, for the Q_i), but these bounds are too disparate to be of much use (more on this in Section 3). The best way to investigate model robustness is usually just to try other possible models. What follows is a minimally parameterized model, which is actually the model we produced when challenged in the paper at the end of the Section 1.2 to analyze the data before reading further. (For fear of overparameterization, it is often helpful to start out by trying very small models.)

Consider the following assumptions:

(i) $Q_i = q^i$;

(ii) $\lambda_i = (1 - \lambda_0)\,\gamma^i e^{-\gamma}/[(1 - e^{-\gamma})\,i!]$ for $i \geq 1$.

Note that this is a three-parameter model, the parameters being $0 \leq q \leq 1$, $0 < \lambda_0 < 1$, and $\gamma > 0$. Our thoughts in choosing this model were (a) independence of effect of hits is a reasonable starting point, and (b) the number of hits might be approximately Poisson, except that some planes may never come under effective fire (for a variety of reasons), so that extra mass at zero hits is to be anticipated. Thus λ_0 was left unrestricted, while the remaining λ_i were given the truncated Poisson distribution. Of course, these assumptions can also be criticized, but they seemed to be a plausible starting point. Note that these assumptions bypass the need to make Wald's assumption (ii), and also will automatically result in decreasing λ_i (except possibly for λ_0, which seemed so likely to be large that monotonization would probably be unnecessary).

Using the fact that

$$\sum_{i=1}^{\infty} q^i \gamma^i/i! = e^{q\gamma} - 1,$$

the likelihood function (1) can be written (under our assumptions and after some algebra) as

$$L(q, \lambda_0, \gamma) = \lambda_0^{X_{01}}(1 - \lambda_0)^{n - X_{01}}\,(e^{\gamma} - 1)^{(X_{01} - n)}$$
$$\times\,(q\gamma)^{\Sigma i X_{i1}}\,(e^{\gamma} - e^{q\gamma})^{(n - \Sigma X_{i1})}.$$

A routine maximum likelihood analysis for the given data yields $\hat{\lambda}_0 = .8$, $\hat{q} = .85$, and $\hat{\gamma} = 1.38$. How well this model fits the data can be seen in Table 1, which presents the estimated λ_i under this model, namely $\lambda_0^* = .8$ and

$$\lambda_i^* = (1 - \lambda_0^*)\,\hat{\gamma}^i e^{-\hat{\gamma}}/[(1 - e^{-\hat{\gamma}})\,i!], \quad i \geq 1,$$

along with the expected observations,

$$\hat{X}_{i1} = n \cdot \hat{P}_{i1} = n \cdot \hat{q}^i \cdot \lambda_i^*,$$

and the actual observations, X_{i1}. For comparison purposes, the unmodeled estimates $\hat{\lambda}_i$ for the λ_i are also given.

The low-parameter model seems to fit the data extremely well. Of course, one would expect to be able to

Table 1. Model Fit

i	λ_i^*	$\hat{\lambda}_i$	$\hat{X}_{i1}$	X_{i1}
0	.8000	.8000	320.0	320
1	.0928	.0940	31.5	32
2	.0640	.0690	18.5	20
3	.0295	.0162	7.2	4
4	.0102	.0095	2.1	2
5	.0028	.0112	0.5	2

fit seven decreasing data points well with some three-parameter model, but not necessarily this well and not necessarily with a model incorporating separate and very specialized structures for the Q_i and the λ_i. In any case, the main feature of interest here is that the answers obtained with this plausible three-parameter model are virtually identical to those of Wald's analysis (especially the $\hat{q}$), so that one can feel somewhat confident about the model robustness of the answers.

Before moving on, it is worthwhile commenting that, instead of the maximum likelihood analysis, a noninformative prior Bayesian analysis could have been performed, using (say) a constant (generalized) prior on the set

$$\Omega = \{(q, \lambda_0, \gamma): \quad 0 \le q \le 1, \quad \lambda_0 \ge \lambda_1, \quad \gamma > 0\}.$$

The advantages of this would be (a) the constraint $\lambda_0 \ge \lambda_1$ is automatically built in; (b) one does not have to worry about having found only local maxima of the likelihood function; and (c) with essentially no extra effort, the posterior variances can be found, yielding good small-sample variance estimates (an attractive alternative to the classical need to resort to large-sample theory).

2. ANALYSIS OF VULNERABILITY AREAS

It is in this aspect of the problem that statistical modeling can reap greater rewards than Wald's approach. Wald needed to assume that the effects of hits on a given area of the aircraft were independent (an assumption that seemed to work reasonably well for the entire aircraft), but this is unlikely to be true for certain vulnerable areas of the aircraft. One obvious example is the important engine area: A multi-engine aircraft might well be able to fly with one engine out, so that the effect of the first hit to the engine area would be inconsequential, while a second hit (to a different engine) could be fatal. It is not hard to think up appropriate models for this situation, and no identifiability problems arise as long as one also makes some effort to model the probability of i hits to a given area (combining, say, the ideas discussed earlier about modeling λ_i with Wald's ideas concerning the probability that a single hit strikes a given area).

3. LOWER BOUNDS ON SURVIVABILITY

A large portion of Wald's analysis is concerned with obtaining lower bounds, Q_i^*, on Q_i, the probability of surviving i hits. One possible use of this would be to allow the aircraft commander to abort a mission if the risk of subsequent hits is too high, but common sense would argue that the relevant factor in such a decision is not how many hits have been sustained (which may even be hard to determine during combat), but rather the amount of actual damage (say, fuel lost or engines destroyed) that can be determined. Data allowing analysis of such occurrences would be hard to come by, and any such analysis would almost certainly involve detailed knowledge about the workings of the aircraft.

A second possible use of the Q_i^* would be in bounding the overall probability of mission survival, presumably for logistic purposes. Clearly

$$\begin{aligned}\Psi &= P\ (\text{survival}) \\ &= \sum_{i=0}^{\infty} Q_i \cdot \lambda_i \\ &\ge \sum_{i=0}^{\infty} Q_i^* \cdot \lambda_i.\end{aligned}$$

The difficulty with this use of the Q_i^* is that Wald determined Q_i^* as $Q_i^* = \min_{\mathcal{P}} Q_i$, where $\mathcal{P}$ is the set of probability structures such that $P_0^*, P_{01}, \ldots, P_{51}$ are equal to the sample proportions. Besides the lack of attention to the effect of sampling error on the analysis, there is the more basic problem that each Q_i is minimized separately over $\mathcal{P}$, and each minimum is attained at a *different* probability structure. Thus

$$\min_{\mathcal{P}} \Psi > \sum_{i=0}^{\infty} Q_i^* \cdot \lambda_i,$$

so that one can get a better lower bound by simply minimizing Ψ directly over $\mathcal{P}$. Of course, this will be computationally more difficult, which could well explain Wald's use of the Q_i^*, but today the additional computation would pose no serious problem.

As a final point, the use of lower bounds at all is probably unwise. Providing one can arrive at model-robust estimates of survivability, use of the estimates discussed in the previous paragraph will generally prove more valuable than use of lower bounds.

4. CONCLUSIONS

All nitpicking aside, the authors seem correct in their conclusion that the answers Wald obtained could not be greatly improved upon today. It can be argued, however, that the methodology employed by Wald was much more difficult and far less flexible than standard methodology involving statistical modeling. Of course, Wald was working under computational limitations (although use of simple statistical models and maximum likelihood methods would not necessarily have been harder computationally), and could perhaps have been writing for a special (nonstatistical) audience. Whatever the reasons for his approach, we can admire his ingenuity while being thankful for the availability of more powerful methods today.

Rejoinder

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1. INTRODUCTION

In this rejoinder we reply to the published remarks of Berger, respond to questions and comments that were raised at the American Statistical Association annual meeting in Toronto in August, 1983, and comment briefly on our recently completed Monte Carlo study on the robustness of Wald's methods.

2. REMARKS ON BERGER'S DISCUSSION

We thank Berger for his thoughtful and thought-provoking commentary on Wald's paper and ours. We are in general agreement with Berger on the main issues he has raised: (a) careful modeling can produce an excellent fit of Wald's data, and the related statistical computations are not that imposing; (b) some of Wald's assumptions are more troublesome than others; and (c) the lower bounds produced by Wald are mathematically interesting but of limited use in decision making. In spite of the consonance of our views with Berger's, there are one or two points on which we differ.

In our Section 3, we described Wald's first data set as an incomplete sample from a multinomial distribution. Berger criticized Wald's assumption that the probability of receiving more than five hits is zero. Actually, the assumption is inconsequential in a multinomial model, since every cell probability associated with an empty cell would be estimated as zero. Thus, Wald's estimator of the parameter q surfaces as the MLE with or without Wald's assumption.

Berger's three-parameter model for Wald's first data set is intriguing. We also tinkered with the Poisson model a bit, but found the fit unacceptable. Berger's idea and rationale for separating the events {0 hits} and {at least one hit} are appealing; it is the kind of idea that seems obvious as soon as it is mentioned, but it is to Berger's credit that he thought of it. Berger mistakenly claims that his model yields decreasing probabilities for 1, 2, 3, . . . hits. Actually, the positive Poisson model with parameter γ has mode $M = \max([\gamma], 1)$, where $[\cdot]$ is the greatest integer function. Thus, these probabilities increase up to M and decrease thereafter. With Wald's data, γ is estimated to be 1.38, so that $\hat{\lambda}_1 > \hat{\lambda}_2 > \hat{\lambda}_3 > \hat{\lambda}_4 > \hat{\lambda}_5$ in this particular application. However, Berger's model does not guarantee this monotonicity. Furthermore, although the Bayesian approach that Berger proposes in order to ensure the inequality $\hat{\lambda}_0 > \hat{\lambda}_1$ can be expanded to cover $\lambda_i > \gamma_{i+1}$ for all i, one should not underestimate the difficulties involved in implementing such an approach in a reasonable manner.

Having pointed out the lack of guaranteed monotonicity of the λ_i's, we hasten to add that, in our view, Berger's model nonetheless has substantial merit. Consider the proposition that $\lambda_1 > \lambda_2$, that is, that an aircraft is more likely to receive one hit than it is to receive two hits. It seems to us that this proposition is not an inviolable imperative. Indeed, the expected number of hits depends quite crucially on the density of fire. Suppose all 400 planes in Wald's first problem were sent on a mission in which intense fire was anticipated. It might well be true that virtually no aircraft would receive only one hit. In fact, it might be that aircraft would be more likely to receive 10 or 12 hits than only one. Berger's model will accommodate such situations, and it should be useful in problems in which the number of hits (to aircraft receiving at least one hit) is expected to have a unimodal distribution. It is interesting that data analysis with the three-parameter model yields the same estimate of q that Wald obtained, which imparts a certain model robustness to Wald's results. One could also interpret this coincidence as speaking to the model robustness of the approach Berger has taken. We are in agreement with the limitations of Wald's results, as discussed by Berger in his Sections 2 and 3.

Motivated in part by Berger's comments on robustness, we conducted our own study on the robustness of Wald's methods. Although the complete details are presented elsewhere (Mangel and Samaniego 1984), we wish to describe our results briefly. We studied two questions: (a) If the assumption that $q_j \equiv q$ for all j is violated, how badly does one do in estimating the p_{i2} using Wald's method? and (b) In the case of unequal q_j, what are the behavior and proper interpretation of Wald's estimator $\hat{q}$? To answer these questions, we carried out a Monte Carlo study in which data in (4.1) were repeatedly generated using a multinomial experiment with parameters $\{p_{ij}\}$ chosen so that the q_j were unequal but had the average $\bar{q} = .851$, as in Wald's data. Our base case involved equal q_j. We measured departure from the true probabilities p_{i2} via a χ^2-like statistic. We found that Wald's model worked very well in a fairly generous neighborhood of the central value $q = .851$, and that the fit was a monotonic function of the dispersion in the set $\{q_1, \ldots, q_5\}$. We also discovered that Wald's estimator $\hat{q}$ is an excellent estimator of the average $\bar{q}$, regardless of the dispersion.

3. COMMENTS AND QUESTIONS RAISED IN TORONTO

A discussant took exception to Wald's derivations and proposed the following alternative analysis. Retaining the

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Applications Section

notation of Section 4 of our article, let

$$p_{j1} = P\{\text{receive exactly } j \text{ hits and survive}\}$$

$$p_{j2} = P\{\text{receive exactly } j \text{ hits and go down}\}.$$

It follows that

$$1 - p_{01} = \sum_{j=1}^{\infty} (p_{j1} + p_{j2}). \quad \text{(R.1)}$$

The following modeling assumption was then introduced (apparently after Wald):

$$p_{j2}/p_{j1} = (1 - q)/q, \quad j = 1, 2, \ldots. \quad \text{(R.2)}$$

Using (R.2) in (R.1) yields

$$1 - p_{01} = \sum_{j=1}^{\infty} p_{j1}\left(1 + \frac{p_{j2}}{p_{j1}}\right)$$

$$= \frac{1}{q}\sum_{j=1}^{\infty} p_{j1}. \quad \text{(R.3)}$$

Thus

$$q = \frac{1}{1 - p_{01}} \sum_{j=1}^{\infty} p_{j1}, \quad \text{(R.4)}$$

leading to the estimator

$$\hat{q} = \frac{1}{1 - a_0} \sum_{j=1}^{\infty} a_j \quad \text{(R.5)}$$

for q. For Wald's data, one obtains $\hat{q} = .75$, which differs from the estimate of .851 obtained by Wald. Further discussion failed to shed any light on the comparative merits of the two estimators.

The confusion during the discussion at Toronto was due in part to blind acceptance of the faulty premise that the two estimators were estimating the same parameter. The proper resolution of this apparent anomaly is that these estimators are not competing against each other, but instead are valid estimators of parameters in different models. Modeling assumption (R.2) is equivalent to

$$p_{j1}/(p_{j1} + p_{j2}) = q, \quad j = 1, 2, \ldots, n, \quad \text{(R.6)}$$

which differs from the modeling assumption

$$p_{j1}/(p_{j1} + p_{j2}) = q^j, \quad j = 1, 2, \ldots, n \quad \text{(R.7)}$$

made by Wald. Indeed, if $f_1, \ldots, f_n$ are continuous, increasing functions mapping (0, 1) onto itself, then the modeling assumption

$$p_{j1}/(p_{j1} + p_{j2}) = f_j(q), \quad j = 1, 2, \ldots, n \quad \text{(R.8)}$$

for the multinomial data in (4.1) gives rise to a unique MLE that can be obtained as the solution of the equation

$$\sum_{j=1}^{n} \frac{a_j}{f_j(q)} = 1 - a_0. \quad \text{(R.9)}$$

Each such model has a parameter q, but the estimator of q in one model has no meaning as an estimator of q in another model.

It remains to comment on the modeling assumptions (R.6) and (R.7). Equations (R.6) constitute the assumption that the chance of surviving another hit, given survival thus far, is always the same. On the other hand, equations (R.7) assert that the conditional probability of surviving another hit, given survival thus far, depends on the number of hits sustained thus far. Wald's general model, with

$$\frac{p_{j1}}{p_{j1} + p_{j2}} = \prod_{i=1}^{j} q_i, \quad j = 1, \ldots, n, \quad \text{(R.10)}$$

stipulates that these conditional probabilities are decreasing. Wald's assumption (R.7) asserts that these probabilities decrease geometrically. It is thus clear that the choice we have discussed is between two models rather than between two estimators. Applications undoubtedly exist in which either one of these models is more appropriate than the other.

A number of people have asked whether Wald's work has actually been used. We do not know whether it was used during World War II, although it was produced early enough in the war to have been available. We do know that during the Vietnam War, analysts at the Operations Evaluation Group of the Center for Naval Analyses used Wald's techniques to study the survivability of the A-4 aircraft. Their analysis led to structural modifications that improved the A-4's survivability. Wald's methods were also used by analysts at Wright Patterson Air Force Base in studying ways of improving the B-52's survivability. Cunningham and Hynd (1946) also provided perspective on the use of statistical analysis during World War II.

One tactical use of this kind of work is the development of rules for exiting from combat. The most important case is the one in which different survival probabilities are estimated (that is, where the q_i are not constant). For example, consider the result presented in Table 1 of our article. The change in the exact value of the probability of surviving i hits as i increases from 1 to 2 is .130, from 2 to 3 is .204, and from 3 to 4 is .235. When confronted with such data, aviators could develop rules of thumb such as, "Stay in combat with up to three hits, but leave after the fourth." Similarly, having an estimate for the survival probabilities would provide the mission planner with one more piece of information that could be used to determine the number of aircraft to send into a particular combat mission.

One factor that Wald did not take into account, but that is quite important, is the crew of the aircraft. Studies done during World War II showed that the crew was an important consideration in determining survivability. For example, crews that had already survived three missions had a much higher probability of continued survival (Morse 1977 discusses this point in more detail).

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